SM311O Second Exam (Solutions)
9 March 2001

1. **(10 Points)** Consider the Laplace equation $\Delta u = 0$. Verify whether $u(x, y) = 2\sinh 3x \sin 3y$ is a solution of this equation.

   Solution: $u(x, y) = 2\sinh 3x \sin 3y$ so $u_x = 6\cosh 3x \sin 3y$, $u_y = 6\sinh 3x \cos 3y$, $u_{xx} = 18\sinh 3x \sin 3y$ and $u_{yy} = -18\sinh 3x \sin 3y$. Therefore, Δu, which by definition equals $u_{xx} + u_{yy}$, vanishes.

2. **(20 points)** Let $\psi = x^2 - y^2 + xy$ be the stream function of a flow.

   (a) Determine the velocity field associated with ψ.

   (b) Determine the circulation of this flow around a circle of radius 2 centered at the origin.

   Solution:

   (a) $\psi = x^2 - y^2 + xy$ so v, which is related to ψ by $v_1 = \frac{\partial \psi}{\partial y}$ and $v_2 = -\frac{\partial \psi}{\partial x}$, takes the form
   $$v_1 = x - 2y, \qquad v_2 = -2x - y.$$

   (b) $\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} bfv_{_C} \cdot \frac{d\mathbf{r}}{dt}\, dt$ where $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$. Hence,
   $$\oint_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} \langle 2\cos t - 2(2\sin t), -2(2\cos t) - 2\sin t\rangle \cdot \langle -2\sin t, 2\cos t\rangle\, dt =$$
   $$-8\int_0^{2\pi} (\cos^2 t + \cos t \sin t - \sin^2 t)\, dt = 0.$$

3. **(20 points)** Let $\mathbf{v} = \langle 2xy, x^2 - 2yz + 2, 1 - y^2\rangle$.

   (a) Determine whether this velocity field has a potential ϕ. If yes, find ϕ.

   (b) Determine the line integral of this flow along the parabola $y = 1 - x^2$ in the $z = 2$ plane from A to B where $A = (1, 0, 2)$ and $B = (2, -3, 2)$.

   solution: $\mathbf{v}\langle 2xy, x^2 - 2yz + 2, 1 - y^2\langle$.

   (a) $\nabla \times \mathbf{v} = \langle 0, 0, 0\rangle$ so $\mathbf{v}$ has a potential ϕ. Since by definition $\nabla\phi = \mathbf{v}$, we have the following relations that define ϕ:
   $$\frac{\partial \phi}{\partial x} = 2xy, \qquad \frac{\partial \phi}{\partial y} = x^2 - 2yz + 2, \qquad \frac{\partial \phi}{\partial z} = 1 - y^2. \tag{1}$$

We start by integrating (1)a with respect to x to get

$$\phi(x, y, z) = x^2 y + f(y, z), \tag{2}$$

where f is the constant of integration. Next, differentiate the above with respect to y to get

$$\frac{\partial \phi}{\partial y} = x^2 + \frac{\partial f}{\partial y}.$$

(**and not** $f'(y, z)$**!**) Compare this result with (1)b and conclude

$$-2yz + 2 = \frac{\partial f}{\partial y}.$$

Integrating this equation yields

$$f(y, z) = -y^2 z + 2y + g(z)$$

where g is the constant of integration. Putting this together with (2) we get

$$\phi = x^2 y - y^2 z + 2y + g(z). \tag{3}$$

Finally, differentiating (3) with respect to z to get $\frac{\partial \phi}{\partial z} = -y^2 + g'(z)$. Compare this result with (1)c to get

$$g'(z) = 1$$

or $g(z) = z$. Hence,

$$\phi = x^2 y - y^2 z + 2y + z.$$

Check: $\nabla \phi = \langle 2xy, x^2 - 2yz + 2, 1 - y^2 \rangle = \mathbf{v}$.

(b) i. **The easy way**: Since $\mathbf{v}$ has a potential

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \phi|_B - \phi|_A = (x^2 y - y^2 z + 2y + z)_{(2,-3,2)} - (x^2 y - y^2 z + 2y + z)_{(1,0,2)} = -36.$$

ii. **The hard way**: Parametrize C as

$$\mathbf{r}(t) = \langle t, 1 - t^2, 2 \rangle.$$

Then

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_1^2 \langle 2t(1-t^2), t^2 - 4(1-t^2) + 2, 1 - (1-t^2)^2 \rangle \cdot \langle 1, -2t, 0 \rangle \, dt = -36.$$

(c) **(20 points)** Let $f(x) = \begin{cases} x & \text{if } 0 \le x < \frac{1}{2}, \\ 1-x & \text{if } \frac{1}{2} \le x < 1. \end{cases}$

i. Find the Fourier sine series of f in the interval $(0,1)$.

ii. Use the first nonzero term of the Fourier series and evaluate it at $x = \frac{1}{2}$. How much does this value differ from $f(\frac{1}{2})$?

solution:

i. Here $L = 1$. So $F = \sum_0^\infty a_n \sin n\pi x$ where

$$a_n = \frac{(f, \sin n\pi x)}{(\sin n\pi x, \sin \pi x)} = 2(f, \sin n\pi x),$$

where the notation $(,)$ means $(f,g) = \int_0^l f(x)g(x)\,dx$. But

$$(f, \sin n\pi x) = \int_0^1 f(x) \sin n\pi x\,dx = \int_0^{\frac{1}{2}} x \sin n\pi x\,dx + \int_{\frac{1}{2}}^1 (1-x) \sin n\pi x\,dx =$$

$$(-\frac{1}{2n\pi} \cos \frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin \frac{n\pi}{2}) +$$
$$(\frac{1}{2n\pi} \cos \frac{n\pi}{2} + \frac{1}{n^2\pi^2} \sin \frac{n\pi}{2} - \frac{1}{n^2\pi^2} \sin n\pi).$$

so

$$a_n = 2(f, \sin n\pi x) = \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}.$$

Finally,

$$F = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \sin n\pi x. \tag{4}$$

ii. Note that $f(\frac{1}{2}) = \frac{1}{2}$. Let $F(x) = \frac{4}{\pi^2} \sin \pi x$ be the first partial sum of f. $F(\frac{1}{2}) = \frac{4}{\pi^2} = 0.405285$. So $f(\frac{1}{2}) - F(\frac{1}{2}) = 0.0947153$.

(d) **(30 points)** Consider the initial-boundary value problem for the heat equation

$$u_t = 9u_{xx}, \qquad u(x,0) = f(x), \qquad u(0,t) = u(1,t) = 0,$$

where f is defined in the previous problem.

i. Explain in words what $u(x,t)$ and each term in the above equations represent.

ii. Start with separation of variables and determine the solution to this problem (you may wish to use the result of your computations in the previous problem).

iii. Using only one term of the series solution in part (a), determine how long ittakes for the temperature at $x = \frac{1}{2}$ to reach 50% of its original temperature.

solution:

i. $u(x,t)$ represents the temperature in a bar of length L at position x at time t. u_t is the rate of change of temperature with repsect to time. u_{xx} is the second derivative of temperature with respect to x. $u(x,0)$ represents the initial temperature in the bar. $u(0,t)$ and $u(1,t)$ represent the boundary temperatures. They also show that the bar has length 1. 9 is a bulk physical constant that encompasses the density, specific heat and the thermal conductivity of the bar.

ii. We start with separation of variables $u(x,t) = F(x)T(t)$. After substituting this expression in $u_t = 9u_{xx}$, dividing by $9FT$, we get

$$\frac{F''}{F} = \frac{T'}{9T} = -\lambda^2.$$

The two resulting differential equations, $F'' + \lambda^2 F = 0$ and $T' + 9\lambda^2 T = 0$, have the general solutions $F(x) = c_1 \sin \lambda x + c_2 \cos \lambda x$ and $T(t) = e^{-9\lambda^2 t}$. So u takes the form

$$u(x,t) = (c_1 \sin \lambda x + c_2 \cos \lambda x)e^{-9\lambda^2 t}.$$

Now $0 = u(0,t) = c_2 e^{-9\lambda^2 t}$ so $c_2 = 0$. So u reduces to

$$u(x,t) = Ae^{-9\lambda^2 t} \sin \lambda x.$$

Next, $0 = u(1,t) = Ae^{-9\lambda^2 t} \sin \lambda$ which results in $\lambda_n = n\pi$. So the general solution for u is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-9n^2\pi^2 t} \sin n\pi x.$$

Since $u(x,0) = f$, where f is the function defined in the previous problem, we have

$$f = \sum_{n=1}^{\infty} A_n \sin n\pi x$$

which implies that $A_n = \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}$ so that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} e^{-9n^2\pi^2 t} \sin n\pi x.$$

iii. The first nonzero term of series that defines u is

$$\frac{4}{\pi^2}e^{-9\pi^2 t}\sin \pi x.$$

The initial temperature at $x = \frac{1}{2}$ is $f(\frac{1}{2}) = \frac{1}{2}$. 50% of this temperature is $\frac{1}{4}$ so we need to find t so that

$$\frac{4}{\pi^2}e^{-9\pi^2 t} = \frac{1}{4}.$$

Solving for t yields 0.00543902.

********* Bonus **********

(10 points) Consider the differential equation

$$-\Delta u = \lambda u. \tag{5}$$

Let $u(x, y) = \sin ax \sin by$. Determine a relationship between λ, a and b for u to be a solution of (5).

Solution (Bonus): Since $u = \sin ax \sin by$ is supposed to be a solution of $-\Delta u = \lambda u$, it must satisfy this equation. But $-\Delta(\sin ax \sin by) = (a^2 + b^2)\sin ax \sin by$. For this expression to equate λu we need to have

$$\lambda = a^2 + b^2.$$